$$\iiint_D f(x,y,z)\,dv$$

$$\text{volume of } D = \iiint_D dv$$

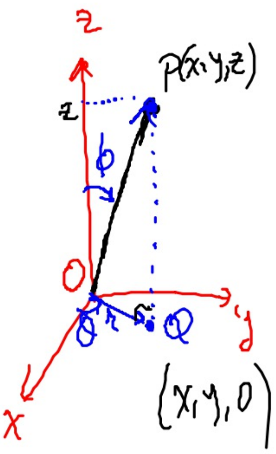
$$dv = dx\,dy\,dz$$
$$dx\,dz\,dy$$
$$dy\,dx\,dz$$
$$dy\,dz\,dx$$
$$dz\,dx\,dy$$
$$dz\,dy\,dx$$

§15.7 cylindrical & spherical

$z = \rho\cos\phi$ | $x = \rho\sin\phi\cos\theta$

$r = \overline{OQ} = \rho\sin\phi$ | $y = \rho\sin\phi\sin\theta$

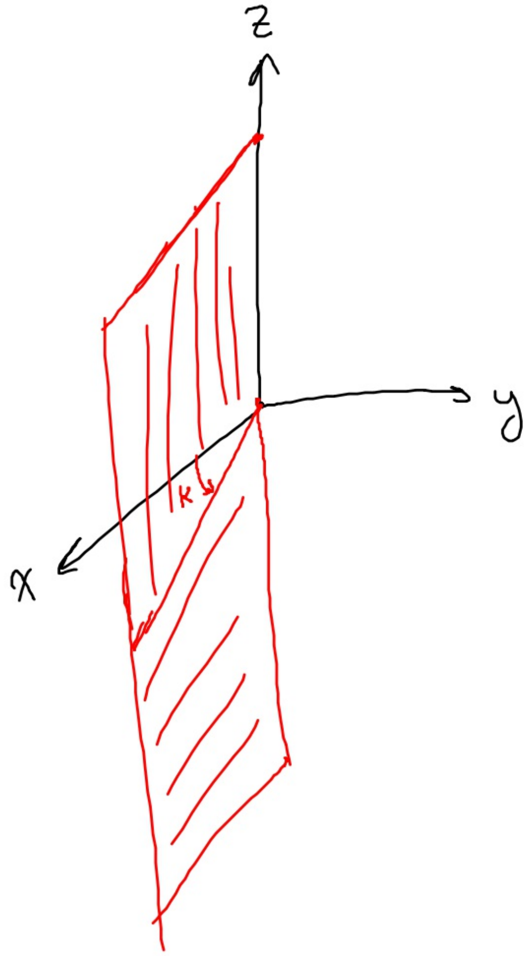
Cartesian	cylindrical	spherical
x, y, z	r, θ, z ($r \geq 0$) $x = r\cos\theta$; $y = r\sin\theta$ $z = z$	ρ, ϕ, θ $\rho = \overline{OP} = \sqrt{x^2+y^2+z^2}$ $\rho^2 = x^2+y^2+z^2$ $\rho^2 = r^2+z^2$ $\rho^2 = r^2+z^2$
$dv = dx\,dy\,dz$	$dv = r\,dr\,d\theta\,dz$	$dv = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$



$0 \leq \phi \leq \pi$

$\theta = K$; half plane ...
hanging on the z-axis

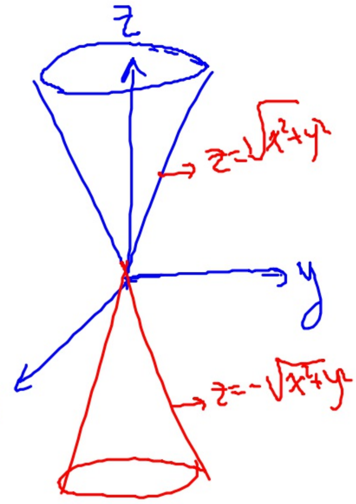
$r = K$; Circular cylinder
around the z-axis.



ϕ	surface
0	Pos. z-axis
$0 < \phi < \frac{\pi}{2}$	Circular cone open up
$\frac{\pi}{2}$	xy-plane
$\frac{\pi}{2} < \phi < \pi$	cone open down
π	Neg. z-axis

(Circular cones around the z-axis with vertices at the origin) examples:

* $z = \sqrt{x^2+y^2} \iff$



$z = r \iff \rho\cos\phi = \rho\sin\phi$

$\sin\phi = \cos\phi$ & $0 \le \phi \le \pi$

$\therefore$ or $\rho = 0$

$\therefore \phi = \frac{\pi}{4}$

* $z^2 = x^2+y^2$

Circular cones (around z-axis)

$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{a^2}$ $\qquad \pm\frac{c}{a} = k$

$z^2 = \frac{c^2}{a^2}(x^2+y^2)$

$z = \pm\frac{c}{a}\sqrt{x^2+y^2}$

$z = k\sqrt{x^2+y^2}$

ex $z = \sqrt{3x^2+3y^2} = \sqrt{3}\, r$

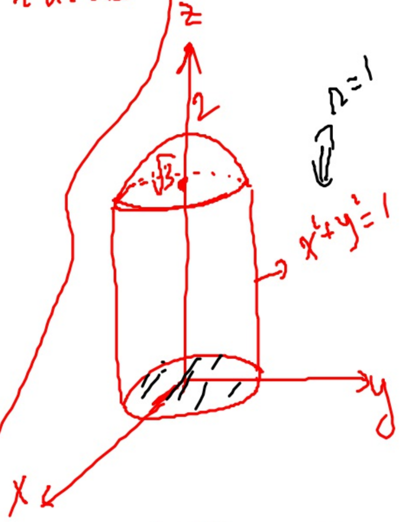
$\rho\cos\phi = \sqrt{3}\,\rho\sin\phi$

11

a) $V = \int_0^{2\pi} \boxed{\int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr} \, d\theta = 2\pi \int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr$

b) $V = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r \, dr \, dz \, d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^{2} \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$

c) $V = \int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \, d\theta \, dz \, dr$



Sphere
$x^2 + y^2 + z^2 = 4$

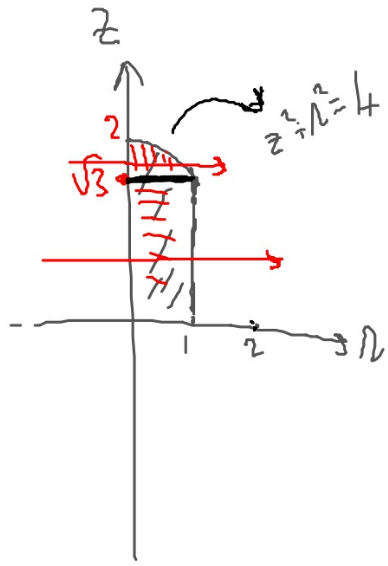
$r^2 + z^2 = 4$

Intersection:

$\left.\begin{array}{l} x^2 + y^2 = 1 \\ x^2 + y^2 + z^2 = 4 \end{array}\right\} \Rightarrow z = \pm\sqrt{3}$

$$\int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr$$

$$= \int_0^{\sqrt{3}} \int_0^1 r \, dr \, dz + \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r \, dr \, dz$$



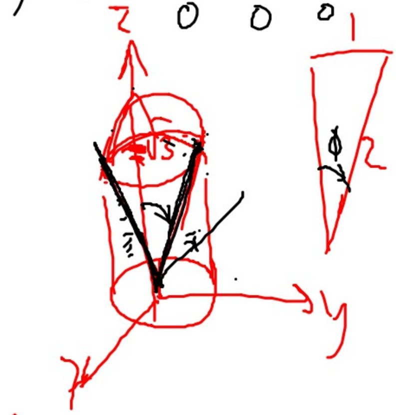
$$z = \sqrt{4-r^2}$$

$$z^2+r^2=4$$

31

a) $$V = \underbrace{\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{cone part}} + \underbrace{\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{\text{outside the cone, inside the cylinder}}$$

b) $$V = \int_0^{2\pi}\int_0^{2}\int_0^{\pi/6} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi}\int_0^{1}\int_{\pi/6}^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$



$$+ \int_0^{2\pi}\int_1^{2}\int_{\pi/6}^{\csc^{-1}\rho} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$

$x^2+y^2=1$

$r=1$

$\rho\sin\phi=1$

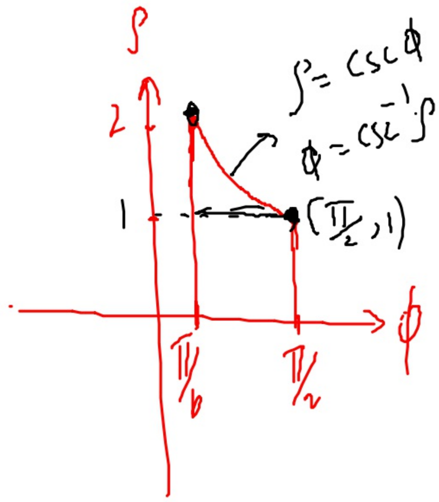
$\rho=\csc\phi$

31.(b)

$$\int_{\pi/6}^{\pi/2} \int_0^{\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi$$

$$= \int_0^1 \int_{\pi/6}^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\rho$$

$$+ \int_1^2 \int_{\pi/6}^{\csc^{-1}\rho} \rho^2 \sin\phi \, d\phi \, d\rho$$

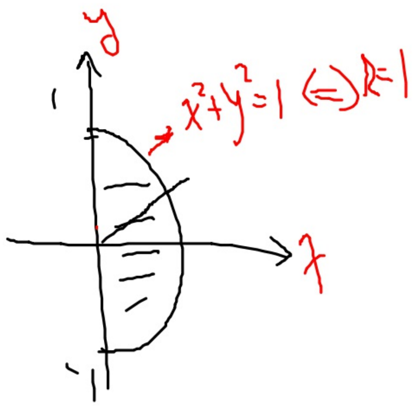


6) $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}(r^2\sin^2\theta + z^2)\,dz\,r\,dr\,d\theta$

$= \dfrac{\pi}{3}$??

#14 $\underbrace{\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}}\boxed{\int_{0}^{x}(x^2+y^2)\,dz}\,dx\,dy$

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^3\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta$$



$$= \underbrace{\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta}_{2}\;\underbrace{\int_{0}^{1} r^4\,dr}_{\frac{1}{5}} = \frac{2}{5}$$

$=$

21) $$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{8}{3}\sin^4\phi \, d\phi \, d\theta = \cdots = \pi^2 \text{ ??}$$

$$\boxed{\begin{aligned}\sin^4\phi &= \left(\frac{1-\cos 2\phi}{2}\right)^2 = \frac{1}{4}\left(1+\cos^2 2\phi - 2\cos 2\phi\right)\\ &= \frac{1}{4}\left(1+\frac{1+\cos 4\phi}{2} - 2\cos 2\phi\right)\\ &= \frac{1}{4}\left(\frac{3}{2}+\frac{\cos 4\phi}{2} - 2\cos 2\phi\right)\\ &= \frac{3}{8}+\frac{\cos 4\phi}{8} - \frac{\cos 2\phi}{2}\end{aligned}}$$

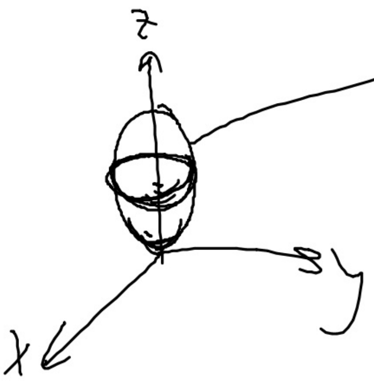
example 3.

$$x^2+y^2+(z-1)^2=1$$

$$x^2+y^2+z^2-2z+1=1$$

$$\rho^2-2\rho\cos\phi=0$$

$$\therefore \rho=0 \text{ or } \boxed{\rho=2\cos\phi}$$



$$x^2+y^2+(z-1)^2=1$$

$$r^2+(z-1)^2=1$$

Recall:

$$r=2\cos\theta$$

$$r^2=2r\cos\theta$$

$$x^2+y^2=2x$$

$$x^2-2x+1+y^2=1$$

$$(x-1)^2+y^2=1$$

37)

$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{8}{3}\cos^3\phi \, \sin\phi \, d\phi \, d\theta$$

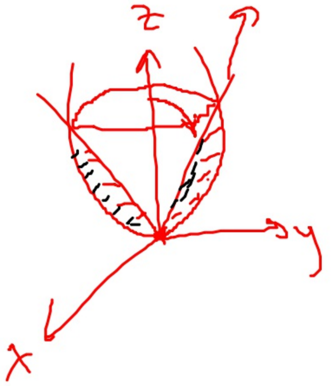
$$= \int_0^{2\pi} \frac{8}{3}\left(-\frac{\cos^4\phi}{4}\right)\Bigg|_{\pi/4}^{\pi/2} d\theta$$

$$= \left(\frac{8}{3}\right)\left(\frac{1}{16}\right)2\pi = \frac{\pi}{3}.$$

top: $\boxed{z = \sqrt{x^2+y^2}}$

Bottom: $\rho = 2\cos\phi$

$\phi = \pi/4$



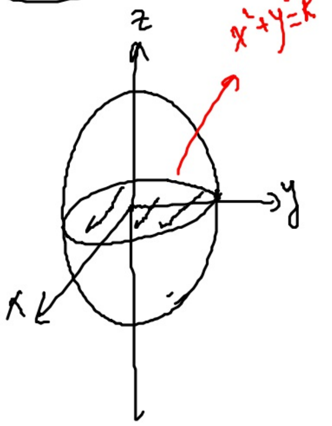
$$z = \sqrt{x^2+y^2} = r$$

$$\rho\cos\phi = \rho\sin\phi$$

$$\phi = \pi/4$$

39 Volume of a sphere with radius R. :

$x^2+y^2+z^2=R^2$
$r^2+z^2=R^2$
$\rho = R$



a) $V = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz\, dy\, dx$

b) $V = \int_{0}^{2\pi} \int_{0}^{R} \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\, dz\, dr\, d\theta$

c) $V = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

d) $v = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

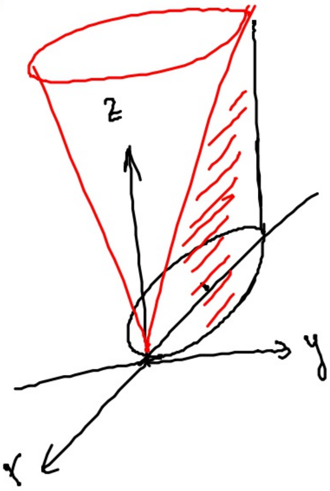
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^{R} \rho^2 d\rho\right)$$

$$= (2\pi)\,(2)\left(\frac{R^3}{3}\right) = \frac{4\pi}{3} R^3 .$$

$$\left(-\cos\phi\right)\Big|_0^{\pi} = \left(\cos\phi\right)\Big|_{\pi}^{0} = 2$$

46 top: $z=\sqrt{x^2+y^2} \Leftrightarrow z=r$

side: $r=-3\cos\theta \Leftrightarrow r^2=-3r\cos\theta \Leftrightarrow x^2+y^2=-3x \Leftrightarrow \left(x+\frac{3}{2}\right)^2+y^2=\frac{9}{4}$



Botto: xy-plane.

$$V=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{-3\cos\theta}\int_0^{r} r\,dz\,dr\,d\theta$$

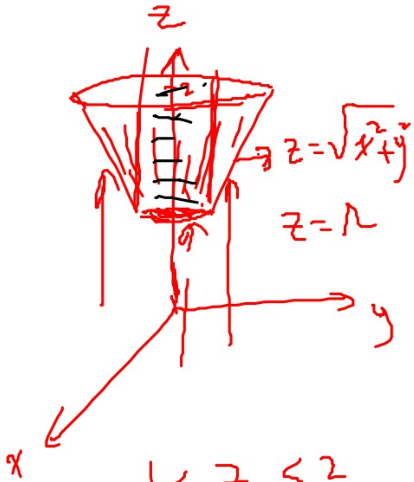
$$=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{-3\cos\theta} r^2\,dr\,d\theta=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left[\frac{r^3}{3}\right]_0^{-3\cos\theta} d\theta$$

$$=-\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 9\cos^3\theta\,d\theta=\cdots=12\ ??$$

52)

$$V = \int_0^{2\pi}\int_0^{1}\int_1^{2} r\,dz\,dr\,d\theta$$

$$+ \int_0^{2\pi}\int_1^{2}\int_r^{2} r\,dz\,dr\,d\theta$$



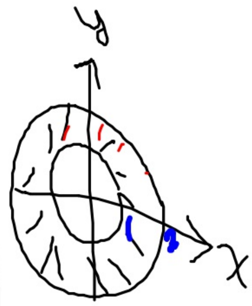
$1 \le z \le 2$

Or

$$V = \int_0^{2\pi}\int_0^{\pi/4 ?}\int_{\sec\phi}^{2\sec\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$z = \rho\cos\phi$

$\rho = z\sec\phi$



$= \cdots = \frac{7\pi}{3}$??

$$\int_{\sec\phi}^{2\sec\phi} \rho^2 \, d\rho = \left(\frac{\rho^3}{3}\right)_{\sec\phi}^{2\sec\phi} = \frac{1}{3}\left(7\sec^3\phi\right)$$

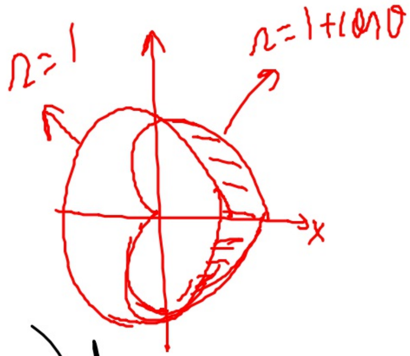
$$= \frac{7}{3} \frac{1}{\cos^3\phi}$$

$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\sec\phi}^{2\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{7}{3} \int_0^{2\pi} \int_0^{\pi/4} \frac{\sin\phi}{\cos^3\phi} \, d\phi \, d\theta = \frac{14\pi}{3} \int_0^{\pi/4} \frac{\sin\phi}{\cos^3\phi} \, d\phi$$

17

$$V = \int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta}\int_{0}^{4} r\,dz\,dr\,d\theta$$



$$= 4\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} r\,dr\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2}\left((1+\cos\theta)^2-1\right)d\theta = 2\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}+2\cos\theta\right)d\theta$$

$$= \ldots\ldots$$